ON L^1-FUNCTIONS WITH A VERY SINGULAR BEHAVIOUR

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Abstract. We give examples of L^1-functions that are essentially unbounded on every non-empty open subset of their domains of definition. We obtain such functions as limits of weighted sums of functions with the unboundedly increasing number of singular points lying at the nodes of standard compressible periodic grids in $\mathbb{R}^n$. Moreover, we prove that the latter (basic) functions possess properties of uniform integral boundedness but do not have a pointwise majorant. Some applications of the main results are given.

Key words and phrases: singularity, local unboundedness, uniform integral boundedness, Γ-compactness.

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1. Introduction

In this article, we give examples of L^1-functions that are essentially unbounded on every nonempty open subset of their domains of definition. We obtain such functions as limits of weighted sums of functions with the unboundedly increasing number of singular points lying at the nodes of standard compressible periodic grids in $\mathbb{R}^n$. Moreover, we prove that the latter (basic) functions possess properties of uniform integral boundedness but do not have a pointwise majorant.

The results obtained allow us to make some important conclusions concerning the conditions under which the Γ-compactness of integral functionals defined on

variable weighted Sobolev spaces was studied in [7,9,10]. However, we think that the main results of the article are of a self-contained interest as well.

2. Functions with singularities at the nodes of periodic grids

Let $n \in \mathbb{N}$, $n \geqslant 2$, and let Ω be a bounded domain of $\mathbb{R}^n$.

For every $y \in \mathbb{R}^n$ and for every $\rho > 0$ we set

$$B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\},$$

and for every $y \in \mathbb{R}^n$ and for every $t \in \mathbb{N}$ we define

$$Q_t(y) = \Big\{x \in \mathbb{R}^n : |x_i - y_i| < \frac{1}{2t},\ i = 1, \dots, n\Big\}.$$

Moreover, for every $t \in \mathbb{N}$ we set

$$Y_t = \{y \in \mathbb{R}^n : ty_i \in \mathbb{Z},\ i = 1, \dots, n\}.$$

We have

$$\forall\, t \in \mathbb{N}, \qquad \bigcup_{y \in Y_t} \overline{Q_t(y)} = \mathbb{R}^n, \tag{2.1}$$

$$\forall\, t \in \mathbb{N},\ \forall\, y, y' \in Y_t,\ y \neq y', \qquad Q_t(y) \cap Q_t(y') = \emptyset. \tag{2.2}$$

The proof of these assertions is simple.

Obviously, for every $t \in \mathbb{N}$ the elements of the set Y_t are the nodes of a periodic grid in $\mathbb{R}^n$ with the period $1/t$. Such standard grids are often used for instance in different constructions to prove Γ-compactness of integral functionals and G-compactness of differential operators with variable domain of definition (see for example [5,6]).

Next, for every $t \in \mathbb{N}$ we set

$$X_t = \{y \in Y_t : Q_t(y) \subset \Omega\}.$$

Owing to (2.1), there exists $m \in \mathbb{N}$ such that for every $t \in \mathbb{N}$, $t > m$, we have $X_t \neq \emptyset$.

For every $t \in \mathbb{N}$, $t > m$, we set

$$G_t = \bigcup_{y \in X_t} B\Big(y, \frac{1}{2t}\Big).$$

It is easy to see that for every $t \in \mathbb{N}$, $t > m$, and for every $y \in X_t$, $B\big(y, \frac{1}{2t}\big) \subset Q_t(y) \subset \Omega$. Therefore, for every $t \in \mathbb{N}$, $t > m$, we have $G_t \subset \Omega$.

Let for every $t \in \mathbb{N}$, $t > m$, $\chi_t : \Omega \to \mathbb{R}$ be the characteristic function of the set G_t, and let for every $t \in \mathbb{N}$, $t > m$, and for every $y \in X_t$, $\chi_{t,y} : \Omega \to \mathbb{R}$ be the characteristic function of the ball $B\big(y, \frac{1}{2t}\big)$.

Further, we denote by $\mathcal{K}$ the set of all functions $\sigma : [0, +\infty) \to (0, +\infty)$ with the properties:

(i) σ is continuous in $(0, +\infty)$;

(ii) $\sigma \geqslant 1$ in $[0, 1]$;

(iii) $\sigma(\rho) \to +\infty$ as $\rho > 0$ and $\rho \to 0$.

(iv) $\displaystyle\int_0^1 \sigma(\rho)\rho^{n-1} d\rho < +\infty$.

For instance if $\sigma_1 : [0, +\infty) \to (0, +\infty)$ is the function such that $\sigma_1(0) = 1$ and $\sigma_1(\rho) = 1/\rho$ for every $\rho > 0$, then $\sigma_1 \in \mathcal{K}$.

For every $\sigma \in \mathcal{K}$ we set

$$M_\sigma = \sigma(1) + n \int_0^1 \sigma(\rho)\rho^{n-1} d\rho.$$

Let us give the following definition: if $\sigma \in \mathcal{K}$ and $t \in \mathbb{N}$, $t > m$, then $\nu_t^{(\sigma)} : \Omega \to \mathbb{R}$ is the function such that for every $x \in \Omega$,

$$\nu_t^{(\sigma)}(x) = \sigma(1)(1 - \chi_t(x)) + \sum_{y \in X_t} \sigma(2t|x - y|)\chi_{t,y}(x).$$

Lemma 2.1. *Let $\sigma \in \mathcal{K}$ and $t \in \mathbb{N}$, $t > m$. Then the following properties hold*:

(i) $\forall\, x \in \Omega \setminus G_t,\ \ \nu_t^{(\sigma)}(x) = \sigma(1)$;

(ii) *if* $y \in X_t$ *and* $x \in B\big(y, \frac{1}{2t}\big)$, *then* $\nu_t^{(\sigma)}(x) = \sigma(2\,t|x-y|)$;

(iii) $\nu_t^{(\sigma)} \geqslant 1$ *in* Ω;

(iv) *the function* $\nu_t^{(\sigma)}$ *is measurable*;

(v) $\nu_t^{(\sigma)} \in L^1(\Omega)$ *and* $\|\nu_t^{(\sigma)}\|_{L^1(\Omega)} \leqslant M_\sigma \mathrm{meas}\,\Omega$.

Proof. Properties (i) and (ii) are immediate consequences of the definition of the function $\nu_t^{(\sigma)}$. From these properties, taking into account that $\sigma \geqslant 1$ in $[0,1]$, we deduce property (iii). Moreover, using properties (i) and (ii) and the continuity of σ in $(0,+\infty)$, we establish that the function $\nu_t^{(\sigma)}$ is continuous in $\Omega \setminus X_t$. Therefore, the function $\nu_t^{(\sigma)}$ is measurable. Thus property (iv) holds.

Next, by property (i), the function $\nu_t^{(\sigma)}$ is summable in $\Omega \setminus G_t$ and

$$\int_{\Omega\setminus G_t} \nu_t^{(\sigma)} dx = \sigma(1)\mathrm{meas}(\Omega \setminus G_t). \tag{2.3}$$

Moreover, taking into account property (ii) and the properties of σ, by means of the change of variables, we establish that for every $y \in X_t$ the function $\nu_t^{(\sigma)}$ is summable in $B\big(y, \frac{1}{2t}\big)$ and

$$\int_{B(y,\frac{1}{2t})} \nu_t^{(\sigma)} dx = \frac{\varkappa_n}{(2t)^n} \int_0^1 \sigma(\rho)\rho^{n-1} d\rho, \tag{2.4}$$

where $\varkappa_n$ is the surface area of the unit sphere of $\mathbb{R}^n$.

Now, taking into account (2.2), we conclude that the function $\nu_t^{(\sigma)}$ is sum-

mable in Ω and, by (2.3) and (2.4),

$$\int_{\Omega} \nu_t^{(\sigma)} dx = \int_{\Omega \setminus G_t} \nu_t^{(\sigma)} dx + \sum_{y \in X_t} \int_{B(y, \frac{1}{2t})} \nu_t^{(\sigma)} dx$$

$$= \sigma(1)\mathrm{meas}(\Omega \setminus G_t) + |X_t| \, \frac{\varkappa_n}{(2t)^n} \int_0^1 \sigma(\rho)\rho^{n-1} d\rho, \quad (2.5)$$

where $|X_t|$ is the number of elements of the set X_t.

From the definition of the set G_t it follows that

$$\mathrm{meas}\, G_t = \frac{|X_t|}{(2t)^n} \,\mathrm{meas}\, B(0,1).$$

This along with the equality $\varkappa_n = n\mathrm{meas}\, B(0,1)$ and (2.5) implies that

$$\int_{\Omega} \nu_t^{(\sigma)} dx = \sigma(1)\mathrm{meas}(\Omega \setminus G_t) + n\biggl(\int_0^1 \sigma(\rho)\rho^{n-1} d\rho\biggr)\mathrm{meas}\, G_t.$$

Hence we get the inequality $\|\nu_t^{(\sigma)}\|_{L^1(\Omega)} \leqslant M_\sigma \mathrm{meas}\, \Omega$. Thus property (v) holds. $\square$

Remark 2.2. If $\sigma \in \mathcal{K}$, $t \in \mathbb{N}$, $t > m$, and $y \in X_t$, then $\nu_t^{(\sigma)}(x) \to +\infty$ as $x \in B\bigl(y, \frac{1}{2t}\bigr) \setminus \{y\}$ and $x \to y$. This follows from property (ii) of Lemma 2.1 and the fact that $\sigma(\rho) \to +\infty$ as $\rho > 0$ and $\rho \to 0$.

Further, for every $\sigma \in \mathcal{K}$ and for every $t \in \mathbb{N}$ we set

$$\mu_t^{(\sigma)} = \sum_{k=1}^{t} k^{-2} \nu_{m+k}^{(\sigma)} \,.$$

Lemma 2.3. *Let $\sigma \in \mathcal{K}$ and $t \in \mathbb{N}$. Then the following properties hold*:

(a) $\mu_t^{(\sigma)} \geqslant 1$ *in* Ω;

(b) $\mu_t^{(\sigma)} < \mu_{t+1}^{(\sigma)}$ *in* Ω;

(c) $\mu_t^{(\sigma)} \in L^1(\Omega)$ *and* $\|\mu_t^{(\sigma)}\|_{L^1(\Omega)} \leqslant 2M_\sigma \mathrm{meas}\, \Omega$.

Proof. Property (a) is a consequence of property (iii) of Lemma 2.1. From the definitions of $\mu_t^{(\sigma)}$ and $\mu_{t+1}^{(\sigma)}$ and property (iii) of Lemma 2.1 we deduce property (b). Finally, property (c) follows from property (v) of Lemma 2.1. $\square$

3. Locally unbounded L^1-functions

We denote by $\mathcal{M}$ the set of all functions $\mu \in L^1(\Omega)$ with the properties:

(i) $\mu \geqslant 1$ in Ω;

(ii) for every nonempty open set $G \subset \Omega$ and for every $C > 0$ there exists a measurable set $H \subset G$ such that $\operatorname{meas} H > 0$ and $\mu \geqslant C$ in H.

Theorem 3.1. *Let $\sigma \in \mathcal{K}$. Then there exists a function $\mu^{(\sigma)} \in \mathcal{M}$ such that*

$$\mu_t^{(\sigma)} \to \mu^{(\sigma)} \quad \textit{a. e. in } \Omega, \tag{3.1}$$

$$\|\mu_t^{(\sigma)}\|_{L^1(\Omega)} \to \|\mu^{(\sigma)}\|_{L^1(\Omega)}, \tag{3.2}$$

$$\forall\, t \in \mathbb{N}, \qquad \mu_t^{(\sigma)} \leqslant \mu^{(\sigma)} \quad \textit{a. e. in } \Omega. \tag{3.3}$$

Proof. By properties (b) and (c) of Lemma 2.3 and B. Levi's theorem (see for instance [3, p. 303]), there exists a function $\tilde{\mu} \in L^1(\Omega)$ such that

$$\mu_t^{(\sigma)} \to \tilde{\mu} \;\; \text{a. e. in } \Omega, \tag{3.4}$$

$$\int_\Omega \mu_t^{(\sigma)} dx \to \int_\Omega \tilde{\mu}\, dx. \tag{3.5}$$

According to (3.4), there exists a set $E \subset \Omega$ with measure zero such that

$$\forall\, x \in \Omega \setminus E, \qquad \mu_t^{(\sigma)}(x) \to \tilde{\mu}(x). \tag{3.6}$$

We define the function $\mu^{(\sigma)} : \Omega \to \mathbb{R}$ by

$$\mu^{(\sigma)}(x) = \begin{cases} \tilde{\mu}(x) & \text{if} \quad x \in \Omega \setminus E, \\ 1 & \text{if} \quad x \in E. \end{cases}$$

Clearly, $\mu^{(\sigma)} \in L^1(\Omega)$. Using (3.6) and property (a) of Lemma 2.3, we establish that $\mu^{(\sigma)} \geqslant 1$ in Ω. Moreover, owing to (3.6), assertion (3.1) holds, and due to (3.5), assertion (3.2) is valid. Besides, by (3.6) and property (b) of Lemma 2.3, assertion (3.3) holds.

Next, let G be a nonempty open set of $\mathbb{R}^n$, $G \subset \Omega$, and let $C > 0$. We fix $z \in G$. Obviously, there exists $\rho_0 > 0$ such that

$$B(z, \rho_0) \subset G. \tag{3.7}$$

We fix $l \in \mathbb{N}$ such that $l > n/\rho_0$. By (2.1), there exists $y \in Y_{m+l}$ such that $z \in \overline{Q_{m+l}(y)}$. Since $n/l < \rho_0$, we have $Q_{m+l}(y) \subset B(z, \rho_0)$. This and (3.7) imply that

$$Q_{m+l}(y) \subset G. \tag{3.8}$$

Therefore, $Q_{m+l}(y) \subset \Omega$. Hence

$$y \in X_{m+l}. \tag{3.9}$$

Since $\sigma \in \mathcal{K}$, we have $\sigma(\rho) \to +\infty$ as $\rho > 0$ and $\rho \to 0$. Then there exists $\delta \in (0, 1)$ such that

$$\forall\, \rho \in (0, \delta), \qquad \sigma(\rho) > (C + 1)l^2. \tag{3.10}$$

We set

$$G' = B\Big(y, \frac{\delta}{2(m+l)}\Big) \setminus \{y\}.$$

Evidently, $G' \subset Q_{m+l}(y)$. From this and (3.8) we get $G' \subset G$.

Let $x \in G'$. We have $2(m + l)|x - y| \in (0, \delta)$. Therefore, by (3.10),

$$\sigma(2(m + l)|x - y|) > (C + 1)l^2. \tag{3.11}$$

Moreover, taking into account (3.9) and using property (ii) of Lemma 2.1, we obtain

$$\nu^{(\sigma)}_{m+l}(x) = \sigma(2(m + l)|x - y|). \tag{3.12}$$

Finally, by the definition of $\mu_l^{(\sigma)}$ and property (iii) of Lemma 2.1, we have

$$\mu_l^{(\sigma)}(x) \geqslant l^{-2}\nu_{m+l}^{(\sigma)}(x). \tag{3.13}$$

From (3.11)–(3.13) we infer that

$$\forall\, x \in G', \qquad \mu_l^{(\sigma)}(x) > C + 1. \tag{3.14}$$

Further, by assertion (3.1) and D. Egoroff's theorem (see for instance [3, p. 287]), there exists a measurable set $\Omega' \subset \Omega$ such that

$$\operatorname{meas}(\Omega \setminus \Omega') \leqslant \frac{1}{2} \operatorname{meas} G', \tag{3.15}$$

$$\mu_t^{(\sigma)} \to \mu^{(\sigma)} \;\text{ uniformly in }\; \Omega'. \tag{3.16}$$

We set $H = G' \cap \Omega'$. Clearly, the set H is measurable and $H \subset G$. Moreover, $G' \subset H \cup (\Omega \setminus \Omega')$. This and (3.15) imply that $\operatorname{meas} G' \leqslant \operatorname{meas} H + \frac{1}{2} \operatorname{meas} G'$. Hence $\operatorname{meas} H > 0$.

According to (3.16), there exists $t_0 \in \mathbb{N}$ such that for every $t \in \mathbb{N}$, $t \geqslant t_0$, and for every $x \in \Omega'$,

$$|\mu_t^{(\sigma)}(x) - \mu^{(\sigma)}(x)| \leqslant 1. \tag{3.17}$$

Let $x \in H$. We fix $t \in \mathbb{N}$, $t \geqslant \max(l, t_0)$. Using (3.17), property (b) of Lemma 2.3 and (3.14), we obtain $\mu^{(\sigma)}(x) \geqslant \mu_t^{(\sigma)}(x) - 1 \geqslant \mu_l^{(\sigma)}(x) - 1 > C$. Thus $\mu^{(\sigma)} \geqslant C$ in H. Now, we conclude that $\mu^{(\sigma)} \in \mathcal{M}$. $\square$

By virtue of Theorem 3.1, the set $\mathcal{M}$ is nonempty. Let us state several propositions describing some properties of this set.

Proposition 3.2. *For every $\lambda > 1$ we have $\mathcal{M} \cap L^\lambda(\Omega) \neq \emptyset$.*

Proof. Let $\lambda > 1$. We fix $\mu \in \mathcal{M}$ and set $\mu_\lambda = \mu^{1/\lambda}$. Since $\mu \in \mathcal{M}$, we have $\mu_\lambda \in \mathcal{M}$. Moreover, due to the definition of μ_λ and the inclusion $\mu \in L^1(\Omega)$, we have $\mu_\lambda \in L^\lambda(\Omega)$. Thus $\mu_\lambda \in \mathcal{M} \cap L^\lambda(\Omega)$. Hence $\mathcal{M} \cap L^\lambda(\Omega) \neq \emptyset$. $\square$

Proposition 3.3. *Let $F:(0,+\infty)\to(0,+\infty)$ be a nondecreasing continuous function, and $F(1)=1$. Let $\sigma\in\mathcal{K}$. Suppose that $\sigma F(\sigma)\notin\mathcal{K}$. Then there exists a function $\mu\in\mathcal{M}$ such that $\mu F(\mu)\notin L^1(\Omega)$.*

Proof. We set $\sigma_*=\sigma F(\sigma)$. Due to the properties of F and the inclusion $\sigma\in\mathcal{K}$, the function σ_* has the following properties: σ_* is continuous in $(0,+\infty)$, $\sigma_*\geqslant 1$ in $[0,1]$ and $\sigma_*(\rho)\to+\infty$ as $\rho>0$ and $\rho\to 0$. Hence, taking into account that $\sigma_*\notin\mathcal{K}$, we obtain that

$$\int_0^1 \sigma_*(\rho)\rho^{n-1}d\rho=+\infty. \tag{3.18}$$

Next, by Theorem 3.1, there exists a function $\mu\in\mathcal{M}$ such that

$$\mu_1^{(\sigma)}\leqslant\mu \quad \text{a.e. in} \quad \Omega. \tag{3.19}$$

Suppose that

$$\mu F(\mu)\in L^1(\Omega). \tag{3.20}$$

Using (3.19), the definition of $\mu_1^{(\sigma)}$ and the fact that F is nondecreasing, we establish that $\nu_{m+1}^{(\sigma)}F(\nu_{m+1}^{(\sigma)})\leqslant\mu F(\mu)$ a.e in Ω. This and (3.20) imply that $\nu_{m+1}^{(\sigma)}F(\nu_{m+1}^{(\sigma)})\in L^1(\Omega)$.

Now, we fix $y\in X_{m+1}$ and for every $\varepsilon\in(0,1)$ set

$$K_\varepsilon=\Big\{x\in\mathbb{R}^n:\frac{\varepsilon}{2(m+1)}<|x-y|<\frac{1}{2(m+1)}\Big\}.$$

Obviously, for every $\varepsilon\in(0,1)$,

$$\int_{K_\varepsilon}\nu_{m+1}^{(\sigma)}F(\nu_{m+1}^{(\sigma)})dx\leqslant\|\nu_{m+1}^{(\sigma)}F(\nu_{m+1}^{(\sigma)})\|_{L^1(\Omega)}. \tag{3.21}$$

On the other hand, using property (ii) of Lemma 2.1, the definition of σ_* and the change of variables, we obtain that for every $\varepsilon\in(0,1)$,

$$\int_{K_\varepsilon}\nu_{m+1}^{(\sigma)}F(\nu_{m+1}^{(\sigma)})dx=\frac{\varkappa_n}{2^n(m+1)^n}\int_\varepsilon^1\sigma_*(\rho)\rho^{n-1}d\rho.$$

This and (3.18) imply that

$$\int_{K_\varepsilon} \nu_{m+1}^{(\sigma)} F(\nu_{m+1}^{(\sigma)})dx \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

However, the result obtained contradicts (3.21). Due to this contradiction, we conclude that inclusion (3.20) does not hold. Thus $\mu F(\mu) \notin L^1(\Omega)$. $\square$

Corollary 3.4. *Let $\lambda > 0$. Then there exists a function $\mu \in \mathcal{M}$ such that $\mu(\ln \mu)^\lambda \notin L^1(\Omega)$.*

Proof. Let $F : (0, +\infty) \to (0, +\infty)$ be the function such that for every $\rho \in (0, +\infty)$, $F(\rho) = [\ln(e - 1 + \rho)]^\lambda$. Clearly, the function F is nondecreasing and continuous, and $F(1) = 1$.

Let $\sigma : [0, +\infty) \to (0, +\infty)$ be the function such that

$$\sigma(\rho) = \begin{cases} \dfrac{4}{\rho^n}\Big(\ln\dfrac{1}{\rho}\Big)^{-1}\Big(\ln\ln\dfrac{1}{\rho}\Big)^{-2} & \text{if} \quad 0 < \rho < e^{-e}, \\ 4e^{ne-1} & \text{if} \quad \rho = 0 \quad \text{or} \quad \rho \geqslant e^{-e}. \end{cases}$$

It is easy to see that the function σ is continuous in $(0, +\infty)$. In addition, we have

$$\forall\, \rho \in (0, e^{-e}), \qquad \sigma(\rho) \geqslant \frac{1}{4\rho}\,. \tag{3.22}$$

Using this fact, we establish that $\sigma > 1$ in $[0, 1]$ and $\sigma(\rho) \to +\infty$ as $\rho > 0$ and $\rho \to 0$. Finally, it is not difficult to verify that

$$\int_0^1 \sigma(\rho)\rho^{n-1}d\rho < +\infty.$$

The described properties of the function σ allow us to conclude that $\sigma \in \mathcal{K}$.

Now, let us show that $\sigma F(\sigma) \notin \mathcal{K}$. In fact, let $\rho \in (0, e^{-e})$. By (3.22) and the definition of F, we have

$$F(\sigma(\rho)) > \Big(\ln\frac{1}{4\rho}\Big)^\lambda. \tag{3.23}$$

Since

$$\ln\frac{1}{4\rho} = -2\ln 2 + \frac{2}{e}\ln\frac{1}{\rho} + \Big(1-\frac{2}{e}\Big)\ln\frac{1}{\rho}$$

and $\ln(1/\rho) > e$, the following inequality holds:

$$\ln\frac{1}{4\rho} > \Big(1-\frac{2}{e}\Big)\ln\frac{1}{\rho}\,. \tag{3.24}$$

Moreover, we observe that

$$\Big(\ln\ln\frac{1}{\rho}\Big)^2 < \frac{4}{\lambda^2}\Big(\ln\frac{1}{\rho}\Big)^{\lambda}. \tag{3.25}$$

From (3.23)–(3.25) we deduce that for every $\rho \in (0, e^{-e})$,

$$\sigma(\rho)F(\sigma(\rho))\rho^{n-1} > \frac{\lambda^2}{\rho}\Big(1-\frac{2}{e}\Big)^{\lambda}\Big(\ln\frac{1}{\rho}\Big)^{-1}.$$

Hence

$$\int_0^1 \sigma(\rho)F(\sigma(\rho))\rho^{n-1}d\rho = +\infty.$$

Therefore, $\sigma F(\sigma) \notin \mathcal{K}$. Then, by Proposition 3.3, there exists a function $\mu \in \mathcal{M}$ such that $\mu F(\mu) \notin L^1(\Omega)$. Hence, taking into account that $\mu \in L^1(\Omega)$ and $\mu F(\mu) \leqslant 2^{\lambda}\mu + 2^{\lambda}\mu(\ln\mu)^{\lambda}$ in Ω, we infer that $\mu(\ln\mu)^{\lambda} \notin L^1(\Omega)$. $\square$

Corollary 3.5. *There exists a function $\mu \in \mathcal{M}$ such that for every $\lambda > 1$, $\mu \notin L^{\lambda}(\Omega)$.*

Proof. By Corollary 3.4, there exists a function $\mu \in \mathcal{M}$ such that

$$\mu\ln\mu \notin L^1(\Omega). \tag{3.26}$$

Let $\lambda > 1$. Since $(\lambda-1)\ln\mu < \mu^{\lambda-1}$ in Ω, we have $\mu\ln\mu < \frac{1}{\lambda-1}\mu^{\lambda}$ in Ω. This and (3.26) imply that $\mu \notin L^{\lambda}(\Omega)$. $\square$

4. An exhaustion property of the domain Ω

In this section, we establish that unions of certain balls connected with all the sets X_t, $t > m$, exhaust the domain Ω. This property is essentially used in Section 5 to study the pointwise behaviour of the functions $\nu_t^{(\sigma)}$.

We set $\alpha = 2^{-n}\mathrm{meas}\, B(0,1)$. Evidently, $\alpha \in (0,1)$.

For every k, $t \in \mathbb{N}$ we set

$$\mathcal{B}_t^{(k)} = \bigcup_{y\in X_{m+t}} B\Big(y, \frac{1}{2(m+t)k}\Big).$$

Clearly, if $k, t \in \mathbb{N}$, we have $\mathcal{B}_t^{(k)} \subset \Omega$.

Proposition 4.1. *Let $k \in \mathbb{N}$. Then for every open set $G \subset \Omega$ we have*

$$\liminf_{t\to\infty} \mathrm{meas}(G \cap \mathcal{B}_t^{(k)}) \geqslant \alpha k^{-n}\mathrm{meas}\, G. \tag{4.1}$$

Proof. Let G be an open set of $\mathbb{R}^n$ such that $G \subset \Omega$. In the case $G = \emptyset$ inequality (4.1) is obvious. Consider the case $G \neq \emptyset$. We fix $\varepsilon > 0$ and for every $j \in \mathbb{N}$ set $G_j = \{x \in G : d(x, \partial G) > 1/j\}$. Clearly, $\mathrm{meas}\, G_j \to \mathrm{meas}\, G$. Therefore, there exists $l \in \mathbb{N}$ such that $G_l \neq \emptyset$ and

$$\mathrm{meas}(G \setminus G_l) \leqslant \varepsilon. \tag{4.2}$$

We fix $t \in \mathbb{N}$ such that $t \geqslant nl$ and set

$$X_t' = \{y \in Y_{m+t} : Q_{m+t}(y) \cap G_l \neq \emptyset\}.$$

Moreover, we denote by q_t the number of elements of the set X_t'. By (2.1), we have $X_t' \neq \emptyset$, and owing to (2.1) and the inequality $n/t \leqslant 1/l$, we get

$$G_l \subset \bigcup_{y\in X_t'} \overline{Q_{m+t}(y)} \subset G. \tag{4.3}$$

This and (2.2) imply that

$$(m+t)^{-n} q_t \leqslant \mathrm{meas}\, G. \tag{4.4}$$

Next, from the obvious inclusion $G \setminus \mathcal{B}_t^{(k)} \subset (G_l \setminus \mathcal{B}_t^{(k)}) \cup (G \setminus G_l)$ and (4.2) we obtain

$$\operatorname{meas}(G \setminus \mathcal{B}_t^{(k)}) \leqslant \operatorname{meas}(G_l \setminus \mathcal{B}_t^{(k)}) + \varepsilon. \tag{4.5}$$

Let us estimate the measure of the set $G_l \setminus \mathcal{B}_t^{(k)}$. First of all we observe that, due to (4.3) and the inclusion $G \subset \Omega$,

$$X'_t \subset X_{m+t}. \tag{4.6}$$

Let $x \in G_l \setminus \mathcal{B}_t^{(k)}$. Since $x \in G_l$, by (4.3), there exists $y \in X'_t$ such that $x \in \overline{Q_{m+t}(y)}$. At the same time $x \notin B\big(y, \frac{1}{2(m+t)k}\big)$. This follows from (4.6) and the fact that $x \notin \mathcal{B}_t^{(k)}$. Thus $x \in \overline{Q_{m+t}(y)} \setminus B\big(y, \frac{1}{2(m+t)k}\big)$, and we conclude that

$$G_l \setminus \mathcal{B}_t^{(k)} \subset \bigcup_{y \in X'_t} \Big[\, \overline{Q_{m+t}(y)} \setminus B\Big(y, \frac{1}{2(m+t)k}\Big)\Big].$$

Hence

$$\operatorname{meas}(G_l \setminus \mathcal{B}_t^{(k)}) \leqslant (1 - \alpha k^{-n})(m+t)^{-n} q_t. \tag{4.7}$$

From (4.5), (4.7) and (4.4) we deduce that $\operatorname{meas}(G \setminus \mathcal{B}_t^{(k)}) \leqslant (1 - \alpha k^{-n}) \operatorname{meas} G + \varepsilon$. Therefore, $\operatorname{meas}(G \cap \mathcal{B}_t^{(k)}) \geqslant \alpha k^{-n} \operatorname{meas} G - \varepsilon$. Hence we get (4.1). $\square$

Corollary 4.2. *Let $k \in \mathbb{N}$. Then for every measurable set $H \subset \Omega$ we have*

$$\liminf_{t \to \infty} \operatorname{meas}(H \cap \mathcal{B}_t^{(k)}) \geqslant \alpha k^{-n} \operatorname{meas} H. \tag{4.8}$$

Proof. Let H be a measurable set of $\mathbb{R}^n$ such that $H \subset \Omega$. We fix $\varepsilon > 0$. Clearly, there exists an open set H' of $\mathbb{R}^n$ such that $H' \subset \Omega$ and

$$\operatorname{meas}(H \setminus H') < \varepsilon, \qquad \operatorname{meas}(H' \setminus H) < \varepsilon. \tag{4.9}$$

By Proposition 4.1, we have

$$\liminf_{t \to \infty} \operatorname{meas}(H' \cap \mathcal{B}_t^{(k)}) \geqslant \alpha k^{-n} \operatorname{meas} H'.$$

This and (4.9) imply that

$$\liminf_{t\to\infty} \operatorname{meas}(H \cap \mathcal{B}_t^{(k)}) \geqslant \alpha k^{-n}(\operatorname{meas} H - \varepsilon) - \varepsilon.$$

Hence we get (4.8). $\square$

The following result describes the above-mentioned exhaustion property of the domain Ω.

Proposition 4.3. *For every $k \in \mathbb{N}$ we have*

$$\operatorname{meas}\Big(\Omega \setminus \bigcup_{t=1}^{\infty} \mathcal{B}_t^{(k)}\Big) = 0. \tag{4.10}$$

Proof. Let $k \in \mathbb{N}$. We set

$$\Phi = \Omega \setminus \bigcup_{t=1}^{\infty} \mathcal{B}_t^{(k)}.$$

By Corollary 4.2, we have

$$\alpha k^{-n} \operatorname{meas} \Phi \leqslant \liminf_{t\to\infty} \operatorname{meas}(\Phi \cap \mathcal{B}_t^{(k)}), \tag{4.11}$$

and from the definition of Φ it follows that for every $t \in \mathbb{N}$, $\Phi \cap \mathcal{B}_t^{(k)} = \emptyset$. The latter fact and (4.11) imply that $\operatorname{meas} \Phi = 0$. Thus equality (4.10) holds. $\square$

Remark 4.4. The exhaustion property described by Proposition 4.3 is an analogue of the exhaustion condition which is assumed in some results of [4,8].

5. Further properties of the functions $\nu_t^{(\sigma)}$

Theorem 5.1. *Let $\sigma \in \mathcal{K}$. Then for almost every $x \in \Omega$ the sequence $\{\nu_{m+t}^{(\sigma)}(x)\}$ is unbounded.*

Proof. For every $k \in \mathbb{N}$ we set

$$\mathcal{B}^{(k)} = \bigcup_{t=1}^{\infty} \mathcal{B}_t^{(k)}.$$

Then we define

$$E_0 = \bigcup_{k=1}^{\infty} (\Omega \setminus \mathcal{B}^{(k)}).$$

From Proposition 4.3 it follows that meas $E_0 = 0$.

Next, we set

$$E_1 = \bigcup_{t=1}^{\infty} X_{m+t}.$$

Clearly, meas $E_1 = 0$.

We fix $x \in \Omega \setminus (E_0 \cup E_1)$. Suppose that

$$\text{the sequence } \{\nu^{(\sigma)}_{m+t}(x)\} \text{ is bounded.} \tag{5.1}$$

Then there exists $M > 0$ such that

$$\forall\, t \in \mathbb{N}, \qquad \nu^{(\sigma)}_{m+t}(x) \leqslant M. \tag{5.2}$$

Since $\sigma \in \mathcal{K}$, we have $\sigma(\rho) \to +\infty$ as $\rho > 0$ and $\rho \to 0$. Therefore, there exists $\delta > 0$ such that

$$\forall\, \rho \in (0, \delta), \qquad \sigma(\rho) > M. \tag{5.3}$$

We fix $j \in \mathbb{N}$ such that $j > 1/\delta$. Since $x \in \Omega \setminus E_0$, we have $x \in \mathcal{B}^{(j)}$. Then there exists $l \in \mathbb{N}$ such that $x \in \mathcal{B}^{(j)}_l$. Hence, taking into account the definition of $\mathcal{B}^{(j)}_l$, we obtain that there exists $y \in X_{m+l}$ such that $x \in B\big(y, \frac{1}{2(m+l)j}\big)$. Therefore, by property (ii) of Lemma 2.1, we have

$$\nu^{(\sigma)}_{m+l}(x) = \sigma(2(m+l)|x-y|). \tag{5.4}$$

Moreover, $2(m+l)|x-y| < 1/j < \delta$. At the same time, due to the fact that $x \notin E_1$, we have $x \neq y$. Thus $2(m+l)|x-y| \in (0,\delta)$. This along with (5.3) and (5.4) implies that $\nu^{(\sigma)}_{m+l}(x) > M$. However, by (5.2), we have $\nu^{(\sigma)}_{m+l}(x) \leqslant M$. The contradiction obtained proves that assertion (5.1) is not valid. Therefore, the sequence $\{\nu^{(\sigma)}_{m+t}(x)\}$ is unbounded. $\square$

Corollary 5.2. *Let $\sigma \in \mathcal{K}$. Then there is no function $\psi : \Omega \to \mathbb{R}$ such that for every $t \in \mathbb{N}$, $\nu^{(\sigma)}_{m+t} \leqslant \psi$ a. e. in Ω.*

Proof. Suppose that there exists a function $\psi : \Omega \to \mathbb{R}$ such that for every $t \in \mathbb{N}$, $\nu^{(\sigma)}_{m+t} \leqslant \psi$ a. e. in Ω. Then there exists a set $E' \subset \Omega$ with measure zero such that

$$\text{for every } x \in \Omega \setminus E' \text{ and for every } t \in \mathbb{N} \text{ we have } \nu^{(\sigma)}_{m+t}(x) \leqslant \psi(x). \tag{5.5}$$

Moreover, by Theorem 5.1, there exists a set $E'' \subset \Omega$ with measure zero such that

$$\text{for every } x \in \Omega \setminus E'' \text{ the sequence } \{\nu^{(\sigma)}_{m+t}(x)\} \text{ is unbounded.} \tag{5.6}$$

Let $x \in \Omega \setminus (E' \cup E'')$. Then, in view of (5.5), the sequence $\{\nu^{(\sigma)}_{m+t}(x)\}$ is bounded. At the same time, by (5.6), the sequence $\{\nu^{(\sigma)}_{m+t}(x)\}$ is unbounded. The contradiction obtained leads to the conclusion required. $\square$

Theorem 5.3. *Let $\sigma \in \mathcal{K}$. Then for every open cube $Q \subset \mathbb{R}^n$ we have*

$$\limsup_{t\to\infty} \int_{Q\cap\Omega} \nu^{(\sigma)}_{m+t}\, dx \leqslant M_\sigma \mathrm{meas}(Q \cap \Omega). \tag{5.7}$$

Proof. Let Q be an open cube of $\mathbb{R}^n$. If $Q \cap \Omega = \emptyset$, inequality (5.7) is evident. Thus we may consider that $Q \cap \Omega \neq \emptyset$.

We have

$$Q = \{x \in \mathbb{R}^n : |x_i - z_i| < a/2, \ \ i = 1, \dots, n\},$$

where $z \in \mathbb{R}^n$ and $a > 0$.

We fix $\varepsilon \in (0, 1)$ and set

$$Q_\varepsilon = \{x \in \mathbb{R}^n : |x_i - z_i| < (1 + \varepsilon)a/2, \ \ i = 1, \dots, n\}.$$

It is easy to see that $Q \subset Q_\varepsilon$ and

$$\mathrm{meas}(Q_\varepsilon \setminus Q) \leqslant (2a)^n n\varepsilon. \tag{5.8}$$

Next, we fix $z' \in Q \cap \Omega$. Clearly, there exists $\rho_0 > 0$ such that

$$B(z', \rho_0) \subset Q \cap \Omega. \tag{5.9}$$

We fix $t \in \mathbb{N}$ such that $t > \max\left\{\frac{n}{\rho_0}, \frac{2}{a\varepsilon}\right\}$ and set

$$\tilde{X}_t = \{y \in X_{m+t} : Q \cap Q_{m+t}(y) \neq \emptyset\}.$$

Observe that $\tilde{X}_t \neq \emptyset$. In fact, by (2.1), there exists $y \in Y_{m+t}$ such that $z' \in \overline{Q_{m+t}(y)}$. Evidently, $Q \cap Q_{m+t}(y) \neq \emptyset$. Moreover, if $x \in Q_{m+t}(y)$, we have $|x - z'| \leqslant |x - y| + |z' - y| < n/t < \rho_0$. This and (5.9) imply that $Q_{m+t}(y) \subset \Omega$. Hence $y \in X_{m+t}$. Now, we may conclude that $y \in \tilde{X}_t$. Therefore, the set $\tilde{X}_t$ is nonempty.

We denote by $\tilde{q}_t$ the number of elements of the set $\tilde{X}_t$. Since

$$\bigcup_{y \in \tilde{X}_t} Q_{m+t}(y) \subset Q_\varepsilon \cap \Omega,$$

using (2.2) and (5.8), we get

$$(m+t)^{-n}\, \tilde{q}_t \leqslant \operatorname{meas}(Q \cap \Omega) + (2a)^n n\varepsilon. \tag{5.10}$$

Further, we set

$$G_t' = (Q \cap \Omega) \setminus G_{m+t}, \qquad G_t'' = \bigcup_{y \in \tilde{X}_t} B\Big(y, \frac{1}{2(m+t)}\Big).$$

It is easy to see that $Q \cap \Omega \subset G_t' \cup G_t''$, $G_t' \subset \Omega$ and $G_t'' \subset \Omega$. Then

$$\int_{Q\cap\Omega} \nu_{m+t}^{(\sigma)}\, dx \leqslant \int_{G_t'} \nu_{m+t}^{(\sigma)}\, dx + \int_{G_t''} \nu_{m+t}^{(\sigma)}\, dx. \tag{5.11}$$

Taking into account property (i) of Lemma 2.1, we get

$$\int_{G_t'} \nu_{m+t}^{(\sigma)}\, dx \leqslant \sigma(1)\operatorname{meas}(Q \cap \Omega), \tag{5.12}$$

and using (2.4) and (5.10), we obtain

$$\int_{G''_t} \nu^{(\sigma)}_{m+t}\, dx = \sum_{y\in\tilde{X}_t} \int_{B(y,\frac{1}{2(m+t)})} \nu^{(\sigma)}_{m+t}\, dx = \frac{\varkappa_n\, \tilde{q}_t}{2^n(m+t)^n} \int_0^1 \sigma(\rho)\rho^{n-1} d\rho$$

$$\leqslant \left(n \int_0^1 \sigma(\rho)\rho^{n-1} d\rho \right) [\,\mathrm{meas}(Q\cap\Omega) + (2a)^n n\varepsilon\,]. \quad (5.13)$$

From (5.11)–(5.13) it follows that

$$\int_{Q\cap\Omega} \nu^{(\sigma)}_{m+t}\, dx \leqslant M_\sigma [\,\mathrm{meas}(Q\cap\Omega) + (2a)^n n\varepsilon\,].$$

Hence we deduce (5.7). $\square$

6. Some applications

Let $p \in (1, n)$. We denote by $\mathcal{N}_p$ the set of all nonnegative functions $\nu : \Omega \to \mathbb{R}$ such that $\nu > 0$ a. e. in Ω, $\nu \in L^1_{\mathrm{loc}}(\Omega)$ and $(1/\nu)^{1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$.

Observe that $\mathcal{M} \subset \mathcal{N}_p$.

In [7,9,10] some weighted Sobolev spaces W_s associated with the exponent p, a weight $\nu \in \mathcal{N}_p$ and a sequence of domains $\Omega_s \subset \Omega$ were considered, and theorems on the Γ-compactness of the sequence of integral functionals $J_s : W_s \to \mathbb{R}$ of the form

$$J_s(u) = \int_{\Omega_s} f_s(x, \nabla u) dx$$

were established.

Here we are not giving the corresponding definitions and statements of the results of the above-mentioned articles. We only point to several things connected with the conditions under which the Γ-compactness of integral functionals J_s was proved.

In the above-mentioned articles, it is supposed that the integrands $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ of the functionals J_s satisfy the following conditions:

(a_1) for every $s \in \mathbb{N}$ and for every $\xi \in \mathbb{R}^n$ the function $f_s(\cdot, \xi)$ is measurable in Ω_s;

(a_2) for every $s \in \mathbb{N}$ and for almost every $x \in \Omega_s$ the function $f_s(x, \cdot)$ is convex in $\mathbb{R}^n$;

(a_3) for every $s \in \mathbb{N}$, for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$,

$$c_1\nu(x)|\xi|^p - \psi_s(x) \leqslant f_s(x, \xi) \leqslant c_2\nu(x)|\xi|^p + \psi_s(x).$$

In the latter condition c_1 and c_2 are positive constants and $\{\psi_s\}$ is a sequence of functions such that

(b_1) for every $s \in \mathbb{N}$, $\psi_s \in L^1(\Omega_s)$ and $\psi_s \geqslant 0$ in Ω_s;

(b_2) for every open cube $Q \subset \mathbb{R}^n$,

$$\limsup_{s\to\infty} \int_{Q\cap\Omega_s} \psi_s \, dx \leqslant \int_{Q\cap\Omega} b \, dx,$$

where $b \in L^1(\Omega)$ and $b \geqslant 0$ in Ω.

According to results of Section 5, there exist sequences that satisfy conditions (b_1) and (b_2) but do not have pointwise majorants. Indeed, the following simple proposition holds.

Proposition 6.1. *Let $\sigma \in \mathcal{K}$. Let $b : \Omega \to \mathbb{R}$ be the function such that for every $x \in \Omega$, $b(x) = M_\sigma$. Let for every $s \in \mathbb{N}$, $\psi_s = \nu^{(\sigma)}_{m+s}$ and $\Omega_s = \Omega$. Then the sequence $\{\psi_s\}$ satisfies conditions* (b_1) *and* (b_2) *but there is no function $\psi : \Omega \to \mathbb{R}$ such that for every $s \in \mathbb{N}$, $\psi_s \leqslant \psi$ a. e. in Ω.*

This result follows from properties (iii) and (v) of Lemma 2.1, Theorem 5.3 and Corollary 5.2.

We note that Proposition 6.1 is of interest to compare conditions (a_1)–(a_3) with conditions under which the Γ-compactness of sequences of integral functionals with degenerate variable integrands and the same domain of integration was established in [1,2].

In [1] it is supposed that the integrands $g_s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the functionals under consideration satisfy conditions of measurability and convexity like (a_1) and (a_2) and the following condition:

$$\begin{aligned}&\text{for every } s \in \mathbb{N}, \text{ for almost every } x \in \mathbb{R}^n \text{ and for every } \xi \in \mathbb{R}^n,\\ &w_s(x)|\xi|^p \leqslant g_s(x,\xi) \leqslant \Lambda w_s(x)(1+|\xi|^p),\end{aligned} \tag{6.1}$$

where $\Lambda > 0$ and $\{w_s\}$ is a sequence of nonnegative functions on $\mathbb{R}^n$ satisfying a uniform Muckenhoupt condition.

In order to compare condition (a_3) with condition (6.1), we give the following example.

Example 6.2. Suppose that $p \geqslant 2$ and all the conditions of Proposition 6.1 are satisfied. Let for every $s \in \mathbb{N}$ the function $f_s : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f_s(x,\xi) = \nu(x)|\xi|^p + (\nu(x))^{(p-1)/p}(\psi_s(x))^{1/p}|\xi|^{p-1}, \quad (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{6.2}$$

It is easy to see that the sequence $\{f_s\}$ satisfies conditions (a_1)–(a_3). At the same time, by Proposition 6.1, the sequence $\{\psi_s\}$ satisfies conditions (b_1) and (b_2).

Assume that there exist $\lambda > 0$ and a sequence of functions $\varphi_s : \Omega \to \mathbb{R}$ such that

$$\begin{aligned}&\text{for every } s \in \mathbb{N}, \text{ for almost every } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^n,\\ &\varphi_s(x)|\xi|^p \leqslant f_s(x,\xi) \leqslant \lambda\varphi_s(x)(1+|\xi|^p).\end{aligned} \tag{6.3}$$

This and the property $\nu > 0$ a. e. in Ω imply that there exists a set $\tilde{E} \subset \Omega$ with measure zero such that

$$\forall\, x \in \Omega \setminus \tilde{E}, \quad \nu(x) > 0, \tag{6.4}$$

$$\begin{aligned}&\text{for every } s \in \mathbb{N}, \text{ for every } x \in \Omega \setminus \tilde{E} \text{ and for every } \xi \in \mathbb{R}^n,\\ &\varphi_s(x)|\xi|^p \leqslant f_s(x,\xi) \leqslant \lambda\varphi_s(x)(1+|\xi|^p).\end{aligned} \tag{6.5}$$

We fix $s \in \mathbb{N}$ and $x \in \Omega \setminus \tilde{E}$. Let $\xi \in \mathbb{R}^n$, $\xi \neq 0$. Using (6.5) and (6.2), we obtain

$$\varphi_s(x)|\xi|^p \leqslant \nu(x)|\xi|^p + (\nu(x))^{(p-1)/p}(\psi_s(x))^{1/p}|\xi|^{p-1} \leqslant 2\nu(x)|\xi|^p + \psi_s(x).$$

Therefore, $\varphi_s(x) \leqslant 2\nu(x) + \psi_s(x)|\xi|^{-p}$. Hence, passing to the limit as $|\xi| \to \infty$, we get

$$\varphi_s(x) \leqslant 2\nu(x). \tag{6.6}$$

Now, let $\xi \in \mathbb{R}^n$, $|\xi| = 1$. Using (6.2), (6.5) and (6.6), we obtain

$$\nu(x)|\xi|^p + (\nu(x))^{(p-1)/p}(\psi_s(x))^{1/p}|\xi|^{p-1} \leqslant \lambda\varphi_s(x)(1 + |\xi|^p) \leqslant 2\lambda\nu(x)(1 + |\xi|^p).$$

Therefore, $(\nu(x))^{(p-1)/p}(\psi_s(x))^{1/p} \leqslant 4\lambda\nu(x)$. Hence, taking into account (6.4), we get $\psi_s(x) \leqslant (4\lambda)^p\nu(x)$.

Thus for every $s \in \mathbb{N}$, $\psi_s \leqslant (4\lambda)^p\nu$ a.e. in Ω. However, this contradicts the fact that, by Proposition 6.1, there is no function $\psi : \Omega \to \mathbb{R}$ such that for every $s \in \mathbb{N}$, $\psi_s \leqslant \psi$ a.e. in Ω. The contradiction obtained proves that there is no $\lambda > 0$ and sequence of functions $\varphi_s : \Omega \to \mathbb{R}$ such that assertion (6.3) holds.

As a result, we conclude that the sequence $\{f_s\}$ satisfies conditions (a$_1$)–(a$_3$) but any extensions g_s of the functions f_s on $\mathbb{R}^n \times \mathbb{R}^n$ do not satisfy condition (6.1). Consequently, the sequence $\{f_s\}$ cannot be considered in the framework of conditions imposed on the integrands of functionals in [1]. The same conclusion concerns conditions imposed on the integrands of functionals in [2].

Finally, mention should be made of the following. In [7,9,10] the Γ-compactness of integral functionals was proved under the assumption that there exists a sequence of nonempty open sets $\Omega^{(k)}$ of $\mathbb{R}^n$ such that for every $k \in \mathbb{N}$, $\overline{\Omega^{(k)}} \subset \Omega^{(k+1)} \subset \Omega$, $\operatorname{meas}(\Omega \setminus \Omega^{(k)}) \to 0$ and for every $k \in \mathbb{N}$ the functions ν and b are bounded in $\Omega^{(k)}$. Evidently, if $\nu \in \mathcal{M}$, the given assumption cannot be realized.

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